

# POSITIVITY OF $|\mathfrak{p}|^a|\mathfrak{q}|^b + |\mathfrak{q}|^b|\mathfrak{p}|^a$

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ABSTRACT. We show that

$$\mathcal{J}_{a,b,n} := \frac{1}{2}(|\mathfrak{p}|^a|\mathfrak{q}|^b + |\mathfrak{q}|^b|\mathfrak{p}|^a)$$

is positive under suitable conditions on the exponents  $a$  and  $b$  and the underlying dimension  $n$ . (Here  $\mathfrak{q}$  is the multiplication by  $x$  and  $\mathfrak{p} := i^{-1}\nabla$ .) Furthermore we show a generalization of the generalized Hardy inequalities for the fractional Laplacians.

## 1. INTRODUCTION

The classical Hardy inequality ([3, Formula (4)])

$$\int_a^\infty \left(\frac{F}{x}\right)^\kappa dx \leq \left(\frac{\kappa}{\kappa-1}\right)^\kappa \int_a^\infty f^\kappa dx$$

with  $F(x) = \int_a^x f(t)dt$  and  $\kappa > 1$  is one of the longest known inequalities allowing to bound the weighted  $L^\kappa$ -norm of a decaying function by the  $L^\kappa$ -norm of its gradient (Hardy [3]). In modern textbooks, see, e.g., Reed and Simon [7, p. 169], this occurs ( $\kappa = 2$ ) as the quantum mechanical uncertainty principle lemma and is written in three dimensions as

$$(1) \quad \int_{\mathbb{R}^3} |\nabla \psi|^2 \geq \frac{1}{4} \int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|^2} dx.$$

This, in turn was generalized by Herbst [4] (see also Yafaev [8] and Frank et al [1]) to fractional Laplacians (see (19)).

In a seemingly different context, the excess charge problem of atoms, Lieb[5] needed

$$(2) \quad |\mathfrak{q}||\mathfrak{p}|^2 + |\mathfrak{p}|^2|\mathfrak{q}| > 0$$

which, however, turned out to be equivalent to the quantum mechanical uncertainty principle. Here  $\mathfrak{p} = -i\nabla$  is the momentum operator and  $\mathfrak{q}$  (multiplication by  $x$ ) is the position operator. Lieb [5] showed in fact, that also

$$(3) \quad |\mathfrak{q}||\mathfrak{p}| + |\mathfrak{p}||\mathfrak{q}| > 0$$

in three dimensions by reducing it to (2).

With the advent of graphene physics, two-dimensional versions of Lieb's inequality became of physical interest which, however, could not simply be reduced to (2). Instead, (3) was directly proven [2].

The purpose of this article is to show, that the positivity of the Jordan product  $\mathcal{J}_{a,b,n} := \frac{1}{2}(|\mathfrak{p}|^a|\mathfrak{q}|^b + |\mathfrak{q}|^b|\mathfrak{p}|^a)$  is in fact a generalization which reduces, for  $b = n - a$  to Hardy inequalities for fractional Laplacians. Here  $a$  and  $b$  are positive constants and  $n$  is the underlying dimension of the appropriate function space.

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## 2. POSITIVITY AND RELATION TO GENERALIZED HARDY INEQUALITIES

Our basic result is the following operator inequality on  $L^2(\mathbb{R}^n)$  for the momentum operator  $\mathbf{p} = -i\nabla$  and the position operator  $\mathbf{q}$  (multiplication by  $x$ ).

**Theorem 1.** *Assume  $n \geq a + b$  and  $\min\{a, b\} \in [0, 2]$ . Then on  $C_0^\infty(\mathbb{R}^n)$*

$$(4) \quad 0 < \mathcal{J}_{a,b,n} := \frac{1}{2}(|\mathbf{p}|^a |\mathbf{q}|^b + |\mathbf{q}|^b |\mathbf{p}|^a).$$

In fact, our proof shows more, namely

$$(5) \quad |\mathbf{q}|^{b/2} \mathcal{H}_{a,n} |\mathbf{q}|^{b/2} \leq \mathcal{J}_{a,b,n}$$

where  $\mathcal{H}_{a,n}$  is the Hardy operator of (19).

As indicated in the introduction, the case  $n = 3$ ,  $a = 2$ , and  $b = 1$  has an important consequence in atomic physics: it is an essential ingredient in bounding the total number of electrons that atoms can bind: the number of electrons that an atom can bind can never exceed twice its nuclear charge. This special case was proven and applied in this context by Lieb [5]. The case  $n = 2$  and  $a = b = 1$  plays a similar role in investigating how many electrons a magnetic quantum dot in a graphene layer can bound and was proved and applied in that context (Handrek and Siedentop [2]).

*Proof.* For the proof we can assume that  $a \leq b$ , since, if not, we use the Fourier transform to exchange the role of  $\mathbf{p}$  and  $\mathbf{q}$ .

We first treat the case, that  $a < 2$ . In this case we follow the strategy of [2] and use the identity (20). Thus, by polarization

$$(6) \quad \begin{aligned} t &:= \frac{1}{2}(\psi, (|\mathbf{p}|^a |\mathbf{q}|^b + |\mathbf{q}|^b |\mathbf{p}|^a) \psi) \\ &= \alpha_{a,n} \Re \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy \frac{(\overline{\psi(x)} - \overline{\psi(y)})(|x|^b \psi(x) - |y|^b \psi(y))}{|x - y|^{n+a}} \\ &= \alpha_{a,n} \Re \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy \frac{2|x|^b |\psi(x)|^2 - (|x|^b + |y|^b) \overline{\psi(x)} \psi(y)}{|x - y|^{n+a}}. \end{aligned}$$

Now, setting  $\psi = g/|\cdot|^{(n+b-a)/2}$  we get

$$(7) \quad \begin{aligned} \frac{t}{\alpha_{a,n}} &= \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy \frac{\frac{|x|^b |g(x)|^2}{|x|^{n+b-a}} + \frac{|y|^b |g(y)|^2}{|y|^{n+b-a}} - \frac{2\Re \overline{g(x)} g(y) |y|^b}{|x|^{(n+b-a)/2} |y|^{(n+b-a)/2}}}{|x - y|^{n+a}} \\ &= \int_{\mathbb{R}^n} \frac{dx}{|x|^n} |g(x)|^2 \int_{\mathbb{R}^n} dy \frac{2|x|^{\frac{n+a-b}{2}} - |x|^{\frac{n+a-b}{2}} |y|^{\frac{-n+a+b}{2}} - |x|^{\frac{n+a+b}{2}} |y|^{\frac{-n+a-b}{2}}}{|x - y|^{n+a}} \\ &\quad + \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy \frac{(|x|^b + |y|^b) |g(x) - g(y)|^2}{2|x|^{(n+b-a)/2} |x - y|^{n+a} |y|^{(n+b-a)/2}}. \end{aligned}$$

At this point we could simply drop the last term, since it is positive. However, with minimal extra effort we estimate the last term using  $|x|^b + |y|^b \geq 2|x|^{b/2}|y|^{b/2}$  and

obtain using (21)

(8)

$$\begin{aligned} t &\geq \alpha_{a,n} \int_{\mathbb{R}^n} dx \frac{|g(x)|^2}{|x|^n} \int_{\mathbb{R}^n} dy \frac{2 - |y|^{\frac{a+b-n}{2}} - |y|^{\frac{-n+a-b}{2}}}{(2|y|)^{\frac{n+a}{2}} \left( \frac{|y|+|y|^{-1}}{2} - \omega \cdot \mathbf{e} \right)^{\frac{n+a}{2}}} + (\psi, |\mathbf{q}|^{b/2} \mathcal{H}_{a,n} |\mathbf{q}|^{b/2} \psi) \\ &= \frac{\alpha_{a,n}}{2^{\frac{n+a}{2}}} \int_{\mathbb{R}^n} dx \frac{|g(x)|^2}{|x|^n} \int_{\mathbb{R}^n} \frac{dy}{|y|^n} \frac{2|y|^{\frac{n-a}{2}} - |y|^{b/2} - |y|^{-b/2}}{\left( \frac{|y|+|y|^{-1}}{2} - \omega \cdot \mathbf{e} \right)^{\frac{n+a}{2}}} + (\psi, |\mathbf{q}|^{b/2} \mathcal{H}_{a,n} |\mathbf{q}|^{b/2} \psi) \\ &= \frac{\alpha_{a,n}}{2^{\frac{n+a}{2}}} \int_{\mathbb{R}^n} dx \frac{|g(x)|^2}{|x|^n} \int_{\mathbb{R}^n} \frac{dy}{|y|^n} \frac{|y|^{\frac{n-a}{2}} + |y|^{-\frac{n-a}{2}} - |y|^{\frac{b}{2}} - |y|^{-\frac{b}{2}}}{\left( \frac{|y|+|y|^{-1}}{2} - \omega \cdot \mathbf{e} \right)^{\frac{n+a}{2}}} + (\psi, |\mathbf{q}|^{b/2} \mathcal{H}_{a,n} |\mathbf{q}|^{b/2} \psi) \\ &> 0 \end{aligned}$$

assuming – in the last line – that  $\psi$  is not identical zero. The positivity, i.e., the last inequality, follows from the positivity of the numerator of the last integral which is a consequence of the fact that the function  $f(\alpha) := r^\alpha + r^{-\alpha}$  is strictly monotone increasing for positive  $r$  and  $n - a \geq b$ .

We now supply the missing case that  $\min\{a, b\} = 2$ . Again we may assume that  $a \leq b$  without loss of generality. An easy calculation shows

$$\begin{aligned} &\frac{1}{2}(|\mathbf{p}|^2 |\mathbf{q}|^b + |\mathbf{q}|^b |\mathbf{p}|^2) \\ &= |\mathbf{q}|^{\frac{b}{2}} \left( |\mathbf{p}|^2 + \frac{1}{2} |\mathbf{q}|^{-\frac{b}{2}} [|\mathbf{p}|^2, |\mathbf{q}|^{\frac{b}{2}}], |\mathbf{q}|^{\frac{b}{2}} |\mathbf{q}|^{-\frac{b}{2}} \right) |\mathbf{q}|^{\frac{b}{2}} \\ &= |\mathbf{q}|^{\frac{b}{2}} \left( |\mathbf{p}|^2 - \frac{b^2}{4} |\mathbf{q}|^{-2} \right) |\mathbf{q}|^{\frac{b}{2}} \\ &\geq |\mathbf{q}|^{\frac{b}{2}} \mathcal{H}_{2,n} |\mathbf{q}|^{\frac{b}{2}} > 0, \end{aligned} \tag{9}$$

because  $b \leq n - 2$ . Since the first inequality is actually an equality in the case  $b = n - 2$ , it shows that our assumption  $a + b \leq n$  is critical, since Herbst's inequalities are sharp.  $\square$

### 3. GROUND STATE REPRESENTATION

The result of the previous section can be viewed as a warmup for the following result.

**Theorem 2.** *Assume  $a, b \in (0, \infty)$  with  $a + b \leq n$ ,  $\min\{a, b\} \in (0, 2)$ , and  $\psi \in C_0^\infty(\mathbb{R}^n)$ . Then*

$$\begin{aligned} (10) \quad &(\psi, (\mathcal{J}_{a,b,n} - L_{a,b,n} |\mathbf{q}|^{b-a}) \psi) = (\psi, |\mathbf{q}|^{\frac{b}{2}} \mathcal{H}_{a,n} |\mathbf{q}|^{\frac{b}{2}} \psi) \\ &+ \alpha_{a,n} \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy \frac{(|x|^{\frac{b}{2}} - |y|^{\frac{b}{2}})^2 |\psi(x)| |x|^\gamma - \psi(y) |y|^\gamma}{2|x|^\gamma |x - y|^{n+a} |y|^\gamma} \end{aligned}$$

where  $\gamma = (n + b - a)/2$  and

$$(11) \quad L_{a,b,n} = 2^a \frac{\Gamma(\frac{n-b+a}{4}) \Gamma(\frac{n+b+a}{4})}{\Gamma(\frac{n+b-a}{4}) \Gamma(\frac{n-b-a}{4})}.$$

**Monotony of  $L_{a,b,n}$ :** Note that  $L_{a,b,n}$  is a strictly monotone decreasing function in  $b$  on the interval  $[0, n - a]$  and vanishes at  $n - a$ . The second claim is obvious, since  $\lim_{x \rightarrow 0^+} \Gamma(x) = 0$ . For the first claim we use the log convexity of the Gamma function (Bohr and Mollerup).

**Sharpness:** Formula (10) implies the inequality

$$(12) \quad \mathcal{J}_{a,b,n} > L_{a,b,n} |\mathbf{q}|^{b-a} + |\mathbf{q}|^{b/2} \mathcal{H}_{a,n} |\mathbf{q}|^{b/2}$$

is strict under the assumptions of the theorem, since the remainder term in (10) vanishes, if and only if  $\psi(x) = c|x|^{-\gamma}$  which is only in  $L^2$  when  $c = 0$ . However, the remainder can be made arbitrarily small by a smooth cut-off tending to infinity.

If  $a = 2$ , equality holds in (12) because of the calculation (9).

*Proof.* Pick  $\gamma := \frac{n+b-a}{2}$ . By Fourier transform of  $|\cdot|^{-\alpha}$  (see (18)), we know that

$$\begin{aligned}
 (13) \quad & (|\psi|^2|x|^\gamma, \mathcal{J}_{a,b,n}|x|^{-\gamma}) \\
 &= \frac{1}{2} \int_{\mathbb{R}^n} d\xi |\xi|^a \left( \overline{(|\psi(x)|^2|x|^\gamma)^\wedge(\xi)} (|x|^{b-\gamma})^\wedge(\xi) \right. \\
 &\quad \left. + \overline{(|\psi(x)|^2|x|^{\gamma+b})^\wedge(\xi)} (|x|^{-\gamma})^\wedge(\xi) \right) \\
 &= \frac{1}{2} \left( \frac{B_{n-(\gamma-b)}}{B_{\gamma-b}} \int_{\mathbb{R}^n} d\xi |\xi|^{a-n+(\gamma-b)} (|\psi(x)|^2|x|^\gamma)^\wedge(\xi) \right. \\
 &\quad \left. + \frac{B_{n-\gamma}}{B_\gamma} \int_{\mathbb{R}^n} d\xi |\xi|^{a-n+\gamma} (|\psi(x)|^2|x|^{\gamma+b})^\wedge(\xi) \right) \\
 &= \frac{1}{2} \left( \frac{B_{a+\gamma-b}B_{n-(\gamma-b)}}{B_{n-a-(\gamma-b)}B_{\gamma-b}} + \frac{B_{a+\gamma}B_{n-\gamma}}{B_{n-a-\gamma}B_\gamma} \right) \int_{\mathbb{R}^n} dx |\psi(x)|^2 |x|^{b-a} \\
 &= 2^a \frac{\Gamma(\frac{n-b+a}{4})\Gamma(\frac{n+b+a}{4})}{\Gamma(\frac{n+b-a}{4})\Gamma(\frac{n-b-a}{4})} \int_{\mathbb{R}^n} dx |\psi(x)|^2 |x|^{b-a} \\
 &= L_{a,b,n} \int_{\mathbb{R}^n} dx |\psi(x)|^2 |x|^{b-a}.
 \end{aligned}$$

(Note that we refrained from doing obvious mollifications.) We have a similar computation for the operator  $|\mathbf{q}|^{\frac{b}{2}}|\mathbf{p}|^a|\mathbf{q}|^{\frac{b}{2}}$ ,

$$\begin{aligned}
 (14) \quad & (|\psi|^2|x|^\gamma, |\mathbf{q}|^{\frac{b}{2}}|\mathbf{p}|^a|\mathbf{q}|^{\frac{b}{2}}|x|^{-\gamma}) \\
 &= \int_{\mathbb{R}^n} d\xi |\xi|^a \overline{(|\psi(x)|^2|x|^{\frac{b}{2}+\gamma})^\wedge(\xi)} (|x|^{\frac{b}{2}-\gamma})^\wedge(\xi) \\
 &= \frac{B_{n-(\gamma-\frac{b}{2})}}{B_{\gamma-\frac{b}{2}}} \int_{\mathbb{R}^n} d\xi |\xi|^{a-n+(\gamma-\frac{b}{2})} (|\psi(x)|^2|x|^{\frac{b}{2}+\gamma})^\wedge(\xi) \\
 &= \frac{B_{a+\gamma-\frac{b}{2}}B_{n-(\gamma-\frac{b}{2})}}{B_{n-a-(\gamma-\frac{b}{2})}B_{\gamma-\frac{b}{2}}} \int_{\mathbb{R}^n} dx |\psi(x)|^2 |x|^{b-a} \\
 &= 2^a \left( \frac{\Gamma(\frac{n+a}{4})}{\Gamma(\frac{n-a}{4})} \right)^2 \int_{\mathbb{R}^n} dx |\psi(x)|^2 |x|^{b-a}.
 \end{aligned}$$

On the other hand, by using (20) and polarization we can compute the above quantities again and obtain

$$\begin{aligned}
 (15) \quad & (|\psi|^2|x|^\gamma, \mathcal{J}_{a,b,n}|x|^{-\gamma}) \\
 &= \frac{1}{2} \alpha_{a,n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{dx dy}{|x-y|^{n+a}} \left\{ (|\psi(x)|^2|x|^\gamma - |\psi(y)|^2|y|^\gamma) (|x|^{b-\gamma} - |y|^{b-\gamma}) \right. \\
 &\quad \left. + (|\psi(x)|^2|x|^{b+\gamma} - |\psi(y)|^2|y|^{b+\gamma}) (|x|^{-\gamma} - |y|^{-\gamma}) \right\} \\
 &= \frac{1}{2} \alpha_{a,n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{dx dy}{|x-y|^{n+a}} \left\{ 2|\psi(x)|^2|x|^b + 2|\psi(y)|^2|y|^b \right. \\
 &\quad \left. - |\psi(x)|^2|x|^\gamma|y|^{b-\gamma} - |\psi(y)|^2|x|^{b-\gamma}|y|^\gamma \right. \\
 &\quad \left. - |\psi(x)|^2|x|^{b+\gamma}|y|^{-\gamma} - |\psi(y)|^2|x|^{-\gamma}|y|^{b+\gamma} \right\}.
 \end{aligned}$$

$$0 \leq |\mathbf{p}|^a |\mathbf{q}|^b + |\mathbf{q}|^b |\mathbf{p}|^a$$

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By (6) and subtraction and addition of  $2\Re\overline{\psi(x)}\psi(y)|y|^b + 2\Re\psi(x)\overline{\psi(y)}|x|^b$  in the above braces we get

$$\begin{aligned}
(16) \quad & (|\psi|^2|x|^\gamma, \mathcal{J}_{a,b,n}|x|^{-\gamma}) \\
&= \alpha_{a,n} \Re \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{dx dy}{|x-y|^{n+a}} (\overline{\psi(x)} - \overline{\psi(y)}) (|x|^b \psi(x) - |y|^b \psi(y)) \\
&\quad + \frac{1}{2} \alpha_{a,n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{dx dy}{|x-y|^{n+a}} \left\{ 2\Re\overline{\psi(x)}\psi(y)|y|^b + 2\Re\psi(x)\overline{\psi(y)}|x|^b \right. \\
&\quad \left. - |\psi(x)|^2|x|^\gamma|y|^{b-\gamma} - |\psi(y)|^2|x|^{b-\gamma}|y|^\gamma \right. \\
&\quad \left. - |\psi(x)|^2|x|^{b+\gamma}|y|^{-\gamma} - |\psi(y)|^2|x|^{-\gamma}|y|^{b+\gamma} \right\} \\
&= (\psi, \mathcal{J}_{a,b,n}\psi) - \alpha_{a,n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\frac{1}{2}(|x|^b + |y|^b) dx dy}{|x|^\gamma|x-y|^{n+a}|y|^\gamma} |\psi(x)|x|^\gamma - \psi(y)|y|^\gamma|^2.
\end{aligned}$$

The sesquilinear form of  $|\mathbf{q}|^{\frac{b}{2}}|\mathbf{p}|^a|\mathbf{q}|^{\frac{b}{2}}$  is

$$\begin{aligned}
(17) \quad & (|\psi|^2|x|^\gamma, |\mathbf{q}|^{\frac{b}{2}}|\mathbf{p}|^a|\mathbf{q}|^{\frac{b}{2}}|x|^{-\gamma}) \\
&= \alpha_{a,n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{dx dy}{|x-y|^{n+a}} (|\psi(x)|^2|x|^{\frac{b}{2}+\gamma} - |\psi(y)|^2|y|^{\frac{b}{2}+\gamma})(|x|^{\frac{b}{2}-\gamma} - |y|^{\frac{b}{2}-\gamma}) \\
&= \alpha_{a,n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{dx dy}{|x-y|^{n+a}} \left\{ |\psi(x)|^2|x|^b + |\psi(y)|^2|y|^b \right. \\
&\quad \left. - 2\Re\overline{\psi(x)}\psi(y)|x|^{\frac{b}{2}}|y|^{\frac{b}{2}} + 2\Re\overline{\psi(x)}\psi(y)|x|^{\frac{b}{2}}|y|^{\frac{b}{2}} \right. \\
&\quad \left. - |\psi(x)|^2|x|^{\frac{b}{2}+\gamma}|y|^{\frac{b}{2}-\gamma} - |\psi(y)|^2|x|^{\frac{b}{2}-\gamma}|y|^{\frac{b}{2}+\gamma} \right\} \\
&= (\psi, |\mathbf{q}|^{\frac{b}{2}}|\mathbf{p}|^a|\mathbf{q}|^{\frac{b}{2}}\psi) - \alpha_{a,n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} dx dy \frac{|\psi(x)|x|^\gamma - \psi(y)|y|^\gamma|^2}{|x|^{\gamma-\frac{b}{2}}|x-y|^{n+a}|y|^{\gamma-\frac{b}{2}}}.
\end{aligned}$$

A combination of the computations (13) to (17) and the ground state representation (21) gives us the desired result.  $\square$

## APPENDIX A. AUXILIARY FACTS

For the reader's convenience we collect some helpful known facts:

**Fourier transforms of powers:** For  $\alpha \in (0, n)$

$$(18) \quad B_\alpha \mathcal{F}(|\cdot|^{-\alpha}) = B_{n-\alpha} |\cdot|^{-n+\alpha}$$

with  $B_\alpha := 2^{\frac{\alpha}{2}} \Gamma(\alpha/2)$  (see, e.g., Lieb and Loss [6, Theorem 5.9])

**Generalized Hardy Inequalities (Herbst [4]):** Assume  $a \in (0, n)$ . Then, on  $H^{a/2}(\mathbb{R}^n)$

$$(19) \quad \mathcal{H}_{a,n} := |\mathbf{p}|^a - 2^a \left[ \frac{\Gamma(\frac{n+a}{4})}{\Gamma(\frac{n-a}{4})} \right]^2 |\mathbf{q}|^{-a} > 0.$$

The inequality is sharp in the sense that there is no smaller constant in front of  $|\mathbf{q}|^{-a}$  which allows this inequality on  $C_0^\infty(\mathbb{R}^n)$ .

Hardy's classical inequality is obtained for  $a = 2$ , Kato's inequality is the case  $a = 1$ .

**Fractional Laplacian:** For  $\psi \in H^{a/2}(\mathbb{R}^n)$

$$(20) \quad (\psi, |\mathbf{p}|^a \psi) = \alpha_{a,n} \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy \frac{|\psi(x) - \psi(y)|^2}{|x-y|^{n+a}}$$

with

$$\alpha_{a,n} = \frac{2^{a-1} \Gamma(\frac{n+a}{2})}{\pi^{n/2} |\Gamma(-\frac{a}{2})|}$$

(Frank et al [1, Formula (3.2)]).

**Ground State transformed Hardy Operator (Frank et al [1]):** For all  $\psi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$

$$(21) \quad (\psi, \mathcal{H}_{a,n} \psi) = \alpha_{a,n} \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy \frac{|\psi(x)|x|^{\frac{n-a}{2}} - \psi(y)|y|^{\frac{n-a}{2}}|^2}{|x|^{\frac{n-a}{2}}|x-y|^{n+a}|y|^{\frac{n-a}{2}}}.$$

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